

Topological Spaces

MATH 348

1. Find all possible topologies on $X = \{a, b\}$.

$$\mathcal{T}_1 = \{\emptyset, X\} \quad \leftarrow \text{indiscrete topology}$$

$$\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$$

$$\mathcal{T}_3 = \{\emptyset, \{b\}, X\}$$

$$\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, X\} \quad \leftarrow \text{discrete topology}$$

2. Find all topologies on $X = \{a, b, c\}$ that contain $\{a\}$ and $\{b\}$ as open sets.

$$\mathcal{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

$$\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$$

$$\mathcal{T}_3 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$$

$$\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \quad \leftarrow \text{discrete topology}$$

3. Give an example of a topology on $\mathbb{R}$ with...

- (a) ...exactly three open sets.

$$\mathcal{T} = \{\emptyset, \{1\}, \mathbb{R}\}$$

- (b) ...exactly five open sets.

$$\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \mathbb{R}\}$$

4. Is it possible for a topological space to have exactly one open set?

Yes: $\{\emptyset\}$

5. Let $X = \mathbb{R}$. Which of the following collections of sets form topologies on X ?

(a) $\mathcal{T}_1 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ Yes

(b) $\mathcal{T}_2 = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ No: consider $\bigcup_{a>0} [a, \infty) = (0, \infty)$

(c) $\mathcal{T}_3 = \{S \subset \mathbb{R} \mid 0 \in S\} \cup \{\emptyset\}$ Yes

(d) $\mathcal{T}_4 = \{S \subset \mathbb{R} \mid S \text{ contains either } 0 \text{ or } 1\} \cup \{\emptyset\}$ No: consider $[0, 0.5] \cap [0.5, 1]$

(e) $\mathcal{T}_5 = \{(a, b) \mid a, b \in \mathbb{R}\}$ No: consider $(0, 1) \cup (2, 3)$

We didn't get to this page on Sept. 17.

Continuous Functions on Topological Spaces

MATH 348

1. Consider the space $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Which of the following functions are continuous in this topology?

(a) $f : X \rightarrow X$ defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$

(b) $g : X \rightarrow X$ defined by $g(a) = b$, $g(b) = a$, and $g(c) = c$

2. State another continuous function $h : X \rightarrow X$ for the topological space in problem #6.

3. Consider the topology $\mathcal{T}_1$ in problem #5. Which of the following functions are continuous?

(a) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x$

(b) $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = -x$

4. Give several examples of continuous functions on the topological space $\mathcal{T}_3$ in problem #5.