

22 Sept. 2026

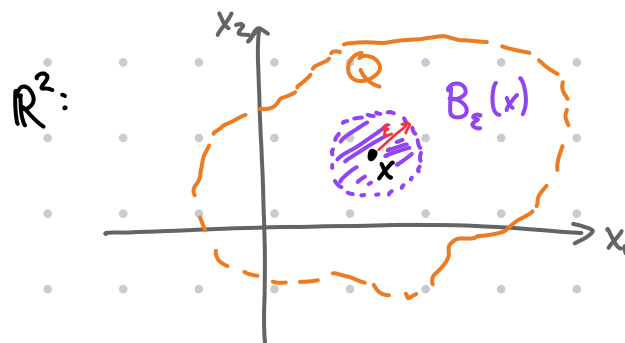
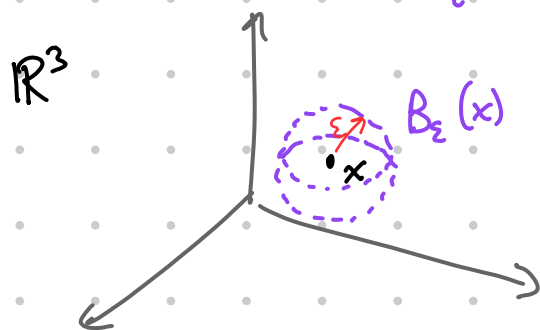
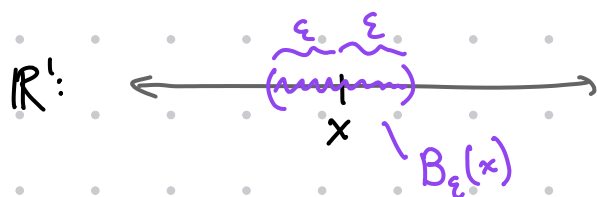
TOPOLOGY DAY 4

RECALL: Topological Space

- T1. X is open
- T2. $\emptyset$ is open
- T3. Arbitrary unions of open sets are open.
- T4. Finite intersections of open sets are open.

Standard topology on $\mathbb{R}^n$:

A subset $Q \subset \mathbb{R}^n$ is open if for every point $x \in Q$, there exists some open ball around x contained in Q .



CONTINUOUS FUNCTIONS ON TOPOLOGICAL SPACES

A function $f: S \rightarrow T$ from one topological space to another is **Continuous** if the preimage $f^{-1}(Q)$ of every open set $Q \subset T$ is open in S .

Continuous Functions on Topological Spaces

MATH 348

1. Consider the space $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Which of the following functions are continuous in this topology?

(a) $f : X \rightarrow X$ defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$

$$f^{-1}(\{b\}) = \{a\} \text{ open}$$

$$f^{-1}(\emptyset) = \emptyset$$

$$f^{-1}(\{a\}) = \{c\} \text{ is not open!}$$

Conclude: f is not continuous!

(b) $g : X \rightarrow X$ defined by $g(a) = b$, $g(b) = a$, and $g(c) = c$

$$g^{-1}(\{b\}) = \{a\}$$

$$g^{-1}(\{a\}) = \{b\}$$

$$g^{-1}(\{a, b\}) = \{a, b\}$$

$$g^{-1}(\emptyset) = \emptyset$$

$$g^{-1}(X) = X$$

2. State another continuous function $h : X \rightarrow X$ for the topological space in problem #6.

• $h : X \rightarrow X$ def $h(a) = c$, $h(b) = c$, $h(c) = c$ ← Constant function

then: $h^{-1}(\{c\}) = \{a, b, c\}$

• Identity map: $\text{id} : X \rightarrow X$ $\text{id}(a) = a$, $\text{id}(b) = b$, $\text{id}(c) = c$

• $k : X \rightarrow X$ def $k(a) = c$, $k(b) = b$, $k(c) = c$

3. Consider the topology $\mathcal{T}_1$ in problem #5. Which of the following functions are continuous?

(a) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x$

$$f^{-1}((a, \infty)) = (\frac{a}{2}, \infty) \text{ open}$$

$$f^{-1}(\emptyset) = \emptyset \quad f^{-1}(\mathbb{R}) = \mathbb{R}$$

$$\mathcal{T}_1 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$



CONTINUOUS!

(b) $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = -x$

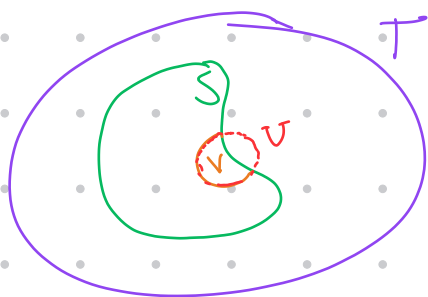
$$g^{-1}((a, \infty)) = (-\infty, -a) \text{ is not open in } \mathcal{T}_1$$

So g is not continuous in this topology!

4. Give several examples of continuous functions on the topological space $\mathcal{T}_3$ in problem #5.

SUBSPACE TOPOLOGY

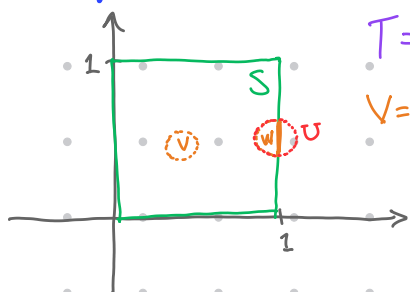
If T is any topological space and $S \subset T$ is any subset, then the **subspace topology** on S takes subsets $V \subset S$ to be open if V is the intersection of S with some open set in T .



Given $V \subset S$:

If there exists U open in T such that $V = U \cap S$, then V is open in S .

Example: $T = \mathbb{R}^2$, $S = [0, 1] \times [0, 1]$



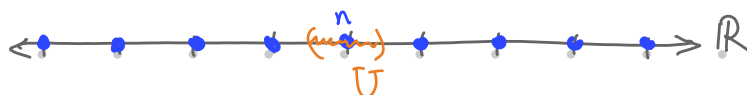
$T = \mathbb{R}^2$ with the standard topology

$V = S \cap V$ so V is open in S
 $\uparrow$ open in T

U is open in T , and

$W = S \cap U$, so
 W is open in S

Example: $T = \mathbb{R}$ with the standard topology
 $S = \mathbb{Z}$ with the subspace topology



$U \cap \mathbb{Z} = \{n\}$ so $\{n\}$ is open in S

With the subspace topology on $\mathbb{Z}$, every set of integers is open.

Subspace Topology

MATH 348

1. Verify that the subspace topology on a subset S of a topological space T is, in fact, a topology.

T1: $S = S \cap T$ and T is open in T , so S is open in the subspace topology

T2: $\emptyset = S \cap \emptyset$ and $\emptyset$ is open in T , so $\emptyset$ is open in the subspace topology

T3: Let $\mathcal{A}$ be a collection of open sets in S . This means that for each $A \in \mathcal{A}$, there is some set $B_A \subset T$, open in T , such that $A = B_A \cap S$. Let $B = \bigcup_{A \in \mathcal{A}} B_A$, which is open in T .

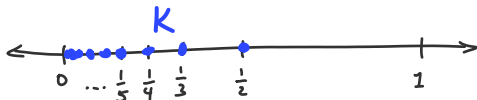
Consider the union $\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (B_A \cap S) = S \cap \left(\bigcup_{A \in \mathcal{A}} B_A \right) = S \cap B$, so $\bigcup_{A \in \mathcal{A}} A$ is open in S .

T4: Let $A_1, A_2, \dots, A_n$ be open in S . This means there exists $B_1, B_2, \dots, B_n$ open in T such that $A_i = S \cap B_i$ for all $i = 1, \dots, n$. Let $B = \bigcap_{i=1}^n B_i$, which is open in T .

Then the intersection $\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n (B_i \cap S) = S \cap \left(\bigcap_{i=1}^n B_i \right) = S \cap B$, so $\bigcap_{i=1}^n A_i$ is open in S .

2. Consider the following subsets of $\mathbb{R}$ with the subspace topology.

- (a) Let $K = \left\{ \frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}_+ \right\}$. Show that the subspace topology on K is the discrete topology.



Let $x \in K$. Then $x = \frac{1}{n}$ for some $n \in \mathbb{Z}^+$.

Now $\frac{1}{n+1}$ is the closest point to x in K .

So let $\varepsilon = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n}$.

The interval $A = (x - \varepsilon, x + \varepsilon)$ is open in $\mathbb{R}$, and $\{x\} = A \cap K$, so $\{x\}$ is open in K with the subspace topology. Since x was arbitrary, each single-element of K is open. This means that every subset of K is open, so the subspace topology on K is the discrete topology.

- (b) Let $K^* = K \cup \{0\}$. Show that the subspace topology on K^* is not the discrete topology.

Note that 0 is in K^* but not in K .

Any open set $A \subset \mathbb{R}$ that contains 0 also contains an open interval $(-\varepsilon, \varepsilon)$ around 0 (where $\varepsilon > 0$ is sufficiently small). Such an open set A must contain $\frac{1}{n}$ for sufficiently large n , so A contains nonzero elements of K^* .

Thus, there is no open set $A \subset \mathbb{R}$ such that $A \cap K^* = \{0\}$.

This means that $\{0\}$ is not open in K^* with the subspace topology, so the subspace topology on K^* is not the discrete topology.

3. Let X be a topological space, and let $Y \subset X$ have the subspace topology.

(a) If A is open in Y , and Y is open in X , show that A is open in X .

We will probably discuss this on Thursday.

(b) If A is closed in Y , and Y is closed in X , show that A is closed in X .