

24 Sept. 2026
 TOPOLOGY DAY 5

From
 Tuesday's
 worksheet:

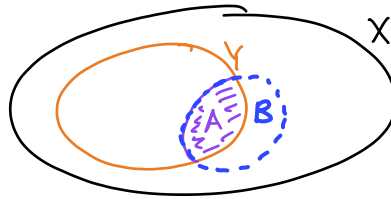
3. Let X be a topological space, and let $Y \subset X$ have the subspace topology.

(a) If A is open in Y , and Y is open in X , show that A is open in X .

Let X be a topological space, with $Y \subset X$ having the subspace topology. Let $A \subset Y$ be open, and $Y \subset X$ be open. We want to show that A is open in X .

By definition of the subspace topology, A open in Y means there exists some open set B in X such that $A = Y \cap B$.

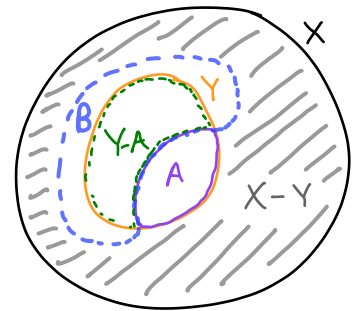
Since Y and B are open in X , it follows that $Y \cap B$ is open in X . Therefore, $A = Y \cap B$ is open in X .



(b) If A is closed in Y , and Y is closed in X , show that A is closed in X .

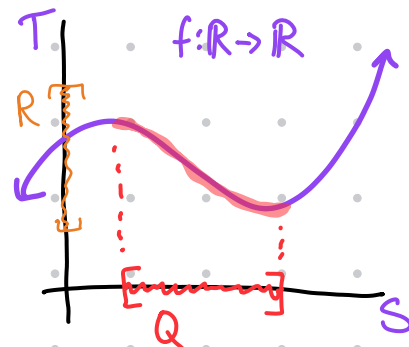
Now let $A \subset Y$ be closed and $Y \subset X$ be closed. Since A is closed in Y , its complement $Y - A$ is open in Y . This means that $Y - A$ is the intersection of Y with some open set $B \subset X$; that is, $Y - A = B \cap Y$.

Since Y is closed in X , its complement $X - Y$ is open in X . Then $B \cup (X - Y)$ is the union of open sets in X , so $B \cup (X - Y)$ is open in X . The only elements of X not in $B \cup (X - Y)$ are the elements of A , since B contains all elements of Y not in A . Thus, $X - A = B \cup (X - Y)$, which means that $X - A$ is open in X . Therefore, A is closed in X .

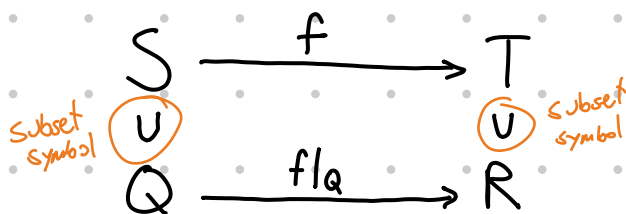


CONTINUITY IN THE SUBSPACE TOPOLOGY

IDEA: If you restrict a continuous function to a subset of its domain, the restriction is still a continuous function.



Proposition 3.35: Let S and T be topological spaces and $f: S \rightarrow T$ be a continuous map. Suppose $Q \subset S$ is a subset whose image, under f , is contained in a subset $R \subset T$, so that f can be restricted to a function $f|_Q: Q \rightarrow R$. If Q and R are given the subspace topologies, then $f|_Q$ is continuous.



BASES FOR A TOPOLOGY

DEFINITION: In a topological space T , a collection $\mathcal{B}$ of open subsets of T is said to form a **basis** for the topology if every open set is the union of some sets in $\mathcal{B}$.

$\mathcal{B}$ contains "basic" open sets; basis $\mathcal{B}$ "generates" the topology

Examples: bases for the standard topology on $\mathbb{R}$

- $B_1 = \{(a,b) \subset \mathbb{R} \mid a,b \in \mathbb{R}\}$ $\leftarrow$ all finite open intervals
- $B_2 = \{(a,b) \subset \mathbb{R} \mid a,b \in \mathbb{Q}\}$ $\leftarrow$ rational endpoint intervals
- $B_3 = \{(a,b) \cup (c,d) \subset \mathbb{R} \mid a < b < c < d \text{ are distinct real numbers}\}$