

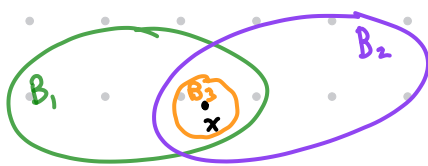
TOPOLOGY DAY 6

29 Sept. 2026

Recall: In a topological space $\mathcal{T}$, a collection $\mathcal{B}$ of open subsets of $\mathcal{T}$ is a basis of the topology if every open subset of $\mathcal{T}$ can be written as a union of sets in $\mathcal{B}$.

THEOREM: Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X . Then $\mathcal{B}$ is the basis for some topology on X if and only if:

- each point of X is in some element of $\mathcal{B}$,
- if B_1 and B_2 are sets in $\mathcal{B}$ and x is a point in $B_1 \cap B_2$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.



DEFINITION OF CONNECTED SPACE

Let X be a topological space.

The space X is **disconnected** if there exists a pair of nonempty, disjoint open sets whose union is X .

In this case, the pair of open sets is called a **separation** of X .

The space X is **connected** if it is not disconnected.

Bases

MATH 348

1. The collection $\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\}$ is a basis for what topology on $\mathbb{R}$?

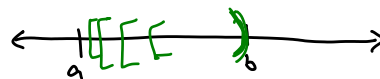
discrete topology

2. The collection $\mathcal{B} = \{[a, b) \subset \mathbb{R} \mid a < b\}$ is a basis for a topology on $\mathbb{R}$. Describe the intervals that are open in this topology.

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

LOWER LIMIT TOPOLOGY

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$



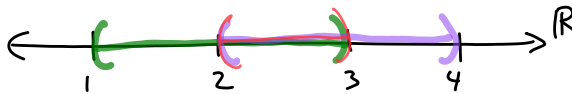
$$\bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$$

3. Let $\mathcal{B} = \{(n, n+1) \mid n \in \mathbb{Z}\}$. Is $\mathcal{B}$ a basis for some topology on $\mathbb{R}$? If so, describe the open sets in this topology. If not, why not?

No, since $\mathbb{R}$ is not any union of elements in $\mathcal{B}$

4. Let $\mathcal{B} = \{(n, n+2) \mid n \in \mathbb{Z}\}$. Is $\mathcal{B}$ a basis for some topology on $\mathbb{R}$? If so, describe the open sets in this topology. If not, why not?

No.



5. Given a set X and a collection $\mathcal{B}$ of subsets of X , what criteria could you use to determine whether or not $\mathcal{B}$ is a basis for a topology on X ?

• If B_1 and B_2 are elements of $\mathcal{B}$, then $B_1 \cap B_2$ is a union of basis elements

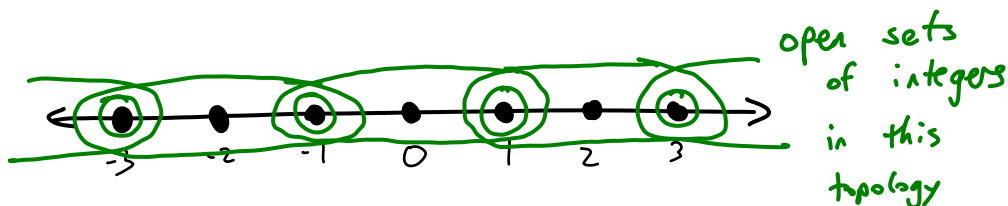
• Union of all elements in $\mathcal{B}$ has to be X .

6. For each $n \in \mathbb{Z}$ define the set

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

Is $\mathcal{B} = \{B(n) \mid n \in \mathbb{Z}\}$ the basis for a topology on $\mathbb{Z}$? — Yes: Digital Line Topology

Give an example of a continuous function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ in this topology. Then give an example of a discontinuous function.

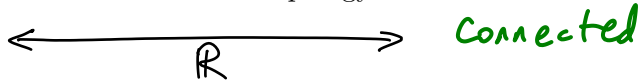


Connectedness

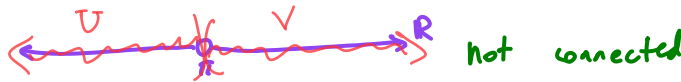
MATH 348

1. Intuitively, which of the following spaces would you say is *connected*?

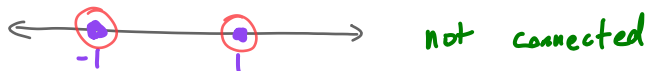
(a) $\mathbb{R}$ with the standard topology



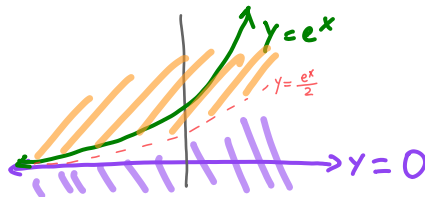
(b) $\mathbb{R} - \{\pi\}$ with the subspace topology



(c) S^0 with the subspace topology



(d) The set $A \subset \mathbb{R}^2$ defined to be the union of the graph of $y = e^x$ and the graph of $y = 0$, with the subspace topology.



$$\tilde{U} = \{(x, y) \in \mathbb{R}^2 \mid y > \frac{e^x}{2}\}$$

an open set in $\mathbb{R}^2$

$$\tilde{V} = \{(x, y) \in \mathbb{R}^2 \mid y < \frac{e^x}{2}\}$$

an open set in $\mathbb{R}^2$

NOTE: To exhibit a separation of A , the sets U and V should be subsets of A .

So denote these regions of $\mathbb{R}^2$ by different symbols such as $\tilde{U}$ and $\tilde{V}$.

2. For each space X in #1 that is disconnected, find a separation of X .

$$U = \tilde{U} \cap A \text{ is the graph of } y = e^x$$

$$V = \tilde{V} \cap A \text{ is the graph of } y = 0$$

The sets U and V are nonempty, disjoint, open in A , and $U \cup V = A$, so U and V form a separation of A

3. Is $\mathbb{R}$ with the discrete topology connected? No

every set is open

example: $U = [-\infty, 0]$ and $V = (0, \infty)$
open in the discrete topology open

U, V disjoint,
 nonempty, open sets
 whose union is $\mathbb{R}$,
 so they form a
 separation of $\mathbb{R}$ in
 the discrete topology

4. Is $\mathbb{Z}$ with the indiscrete topology connected?

only open sets Yes
 are $\emptyset$ and $\mathbb{Z}$

We will return to these problems next time.

5. Prove that topological space X is connected if and only if there are no nonempty proper subsets of X that are both open and closed in X .

6. Let X be a connected topological space. Show that there is no continuous surjective map from X to S^0 .

7. Prove that if X is connected and $f : X \rightarrow Y$ is continuous, then $f(X)$ is connected in Y .