

TOPOLOGY DAY 7

1 October 2026

5. Prove that topological space X is connected if and only if there are no nonempty proper subsets of X that are both open and closed in X .

Equivalently: X is disconnected if and only if there exists a nonempty proper subset of X that is both open and closed.

PROOF:

$\Rightarrow$ Suppose X is disconnected. Then $X = U \cup V$, where U and V are disjoint nonempty open sets. Since X is a disjoint union of U and V , they are complements of each other, which means U and V are also closed.

$\Leftarrow$ Suppose $U \subset X$ is a nonempty proper subset that is both open and closed. Then $V = X - U$ is open, nonempty, and disjoint from U . Thus, U and V form a separation of X , so X is disconnected.

6. Let X be a connected topological space. Show that there is no continuous surjective map from X to S^0 .

Suppose $f: X \rightarrow S^0$ is continuous and surjective. Consider $U = f^{-1}(\{1\})$ and $V = f^{-1}(\{-1\})$. Then U and V are:

- disjoint, since no $x \in X$ can map to both $+1$ and -1 ;
- open, since f is continuous and $\{1\}$ and $\{-1\}$ are open in S^0 ;
- nonempty, since f is surjective

Furthermore, $U \cup V = X$, since $f(x) = \pm 1$ for all $x \in X$.

Thus, U and V form a separation of X , which is a contradiction

Therefore, no such function f exists.

7. Prove that if X is connected and $f: X \rightarrow Y$ is continuous, then $f(X)$ is connected in Y .

Proof by contradiction: Suppose $f(X)$ is disconnected in Y .

Then there are sets U and V that are nonempty, disjoint, open in Y , and whose union is $f(X)$. Consider $f^{-1}(U)$ and $f^{-1}(V)$. These preimages are:

- disjoint
- open
- nonempty

Also, $f^{-1}(U) \cup f^{-1}(V) = X$, so $f^{-1}(U)$ and $f^{-1}(V)$ separate X , which is a contradiction. Thus, $f(X)$ must not be disconnected.

Recall: A topological space X is disconnected if it is the union of two nonempty, disjoint, proper open subsets.



If X is not disconnected, then X is connected.

Motivation for compactness:

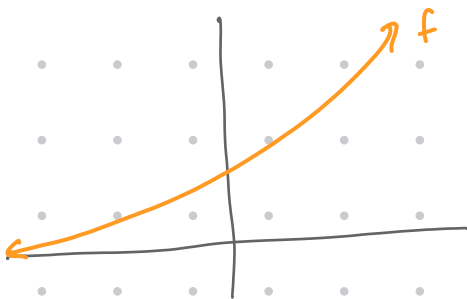
What properties are preserved by continuous functions?

Suppose f is continuous.

If U is open, $f(U)$ might not be open.

If U is closed, $f(U)$ might not be closed.

→ example: $f: \mathbb{R} \rightarrow \mathbb{R}$ defined $f(x) = e^x$



$$f(\mathbb{R}) = (0, \infty)$$

$$f((-\infty, 0]) = (0, 1]$$

Definitions:

An **open cover** of a topological space T is a collection of open subsets of T such that every point of T lies in at least one of the open sets.

A topological space T is **compact** if every open cover of T has a **finite refinement**. This means that it is

possible to select a finite subcollection of the open cover that covers J .

Heine-Borel Theorem: A subspace T of $\mathbb{R}^n$ is compact if and only if T is closed and bounded.

Prop. 4.27: If $f: S \rightarrow T$ is continuous and S is compact, then the image of f is compact.

Compactness

MATH 348

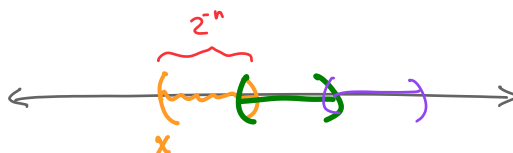
1. Let $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{Z}\}$.

- (a) Does $\mathcal{C}$ cover $\mathbb{R}$? *Yes*
- (b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$? *No*
- (c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$? *Yes*

2. Let $\mathcal{C} = \{(n, \infty) \mid n \in \mathbb{Z}\} \cup \{(-\infty, n) \mid n \in \mathbb{Z}\}$.

- (a) Does $\mathcal{C}$ cover $\mathbb{R}$? *Yes*
- (b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$? *Yes: e.g. $\{(-1, \infty), (-\infty, 1)\}$*
- (c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$? *Yes*

3. Let $\mathcal{C} = \{(x, x + 2^{-n}) \mid x \in \mathbb{R}, n \in \mathbb{Z}_+\}$.



- (a) Does $\mathcal{C}$ cover $\mathbb{R}$? *Yes*
- (b) Does any finite subcollection of $\mathcal{C}$ cover $\mathbb{R}$? *No*
- (c) Does any finite subcollection of $\mathcal{C}$ cover $[0, 10]$? *Yes*

4. Prove that any finite topological space is compact.

Can you see how to prove this?

We will return to this next.

5. Let $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$. Is A compact in $\mathbb{R}$?

6. Is $(0, 1]$ compact as a subspace of $\mathbb{R}$?

7. Is $[0, 1]$ compact as a subspace of $\mathbb{R}$?

8. **Challenge Problem:** Let the set of rationals $\mathbb{Q}$ have the subspace topology from $\mathbb{R}$. Find a set $S \subset \mathbb{Q}$ that is closed and bounded but not compact.