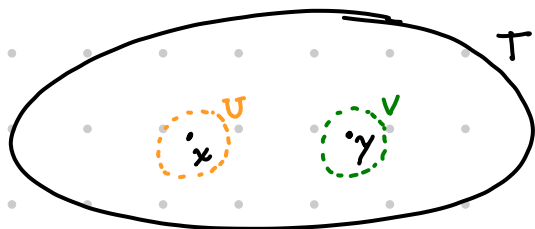


MATH 348 Topology

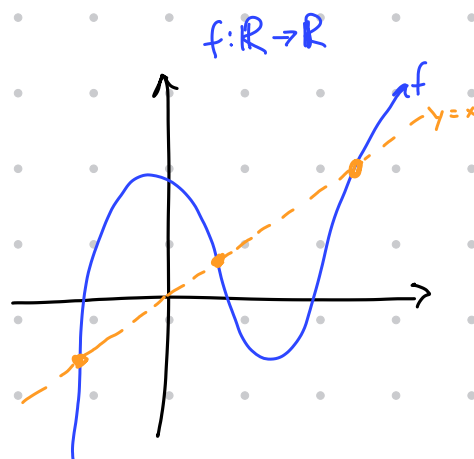
Day 10 • 15 Oct. 2026

DEFINITION: A topological space is **Hausdorff** if, for any two distinct points $x, y \in T$, there are open sets U, V of T such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.



FIXED-POINT SET: If $f: T \rightarrow T$ is a continuous map, then $\text{Fix}(f) = \{x \in T \mid f(x) = x\}$.

Example:

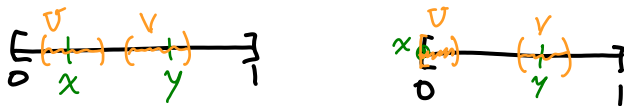


The Hausdorff Property

MATH 348

1. Which of the following spaces are Hausdorff spaces?

(a) $[0, 1]$ with the subspace topology from $\mathbb{R}$ Yes



(b) $\mathbb{R}^n$ with the standard topology Yes



(c) $\{a, b\}$ with the indiscrete topology No

Only open sets: $\emptyset, \{a, b\}$

(d) Any set X with the discrete topology

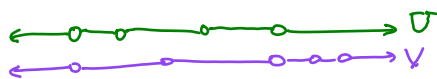
Yes: each point x is in its own open set $\{x\}$

(e) Set $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$

No: choose a and b .

No open set U, V such that $a \in U, b \in V, U \cap V = \emptyset$

(f) $\mathbb{R}$ with the finite complement topology: the open sets are subsets $U \subset \mathbb{R}$ such that $\mathbb{R} - U$ is either finite or all of $\mathbb{R}$



No: For any nonempty open sets U, V in this topology, $U \cap V \neq \emptyset$.

2. Let T be a topological space. Suppose A is a subset of T such that for every $x \in A$ there exists open set U such that $x \in U \subset A$. Prove A is open.

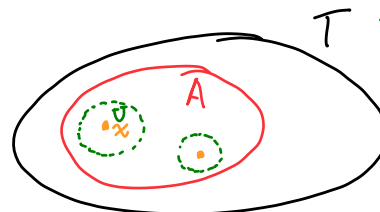
proof: For each $x \in A$, let U_x be an open set such that $x \in U_x \subset A$.

Consider $\bigcup_{x \in A} U_x$, which is open.

Since $U_x \subset A$ for all $x \in A$, $\bigcup_{x \in A} U_x \subset A$.

For any $x \in A$, $x \in U_x \subset \bigcup_{x \in A} U_x$, so $A \subset \bigcup_{x \in A} U_x$.

Therefore, $\bigcup_{x \in A} U_x = A$. So A is open.



This is important!
We should give it a name! How about the "pointwise openness criterion"?

3. Prove that if T is a Hausdorff space, then every single-point subset of T is closed.

Let $x \in T$. We want to show that $T - \{x\}$ is open.

Let $y \in T - \{x\}$. Since T is Hausdorff, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. This implies $x \notin V$, so $V \subset T - \{x\}$.

So every point $y \in T - \{x\}$ is contained in some open subset of $T - \{x\}$. Thus, by #2 above, $T - \{x\}$ is open so $\{x\}$ is closed.

[side note:
 $y \in V \subset T - \{x\}$]

4. If T is a topological space and $f : T \rightarrow T$ is a continuous map, then the **fixed-point set** of f is defined $\text{Fix}(f) = \{x \in T \mid f(x) = x\}$.

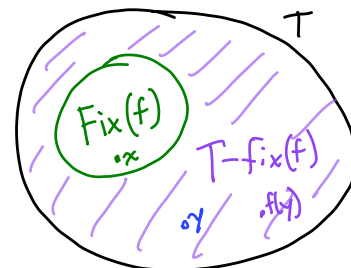
Prove that if T is a Hausdorff space then $\text{Fix}(f)$ is a closed subset of T .

We will show that $T - \text{Fix}(f)$ is open. Let $y \in T - \text{Fix}(f)$.

Then $y \neq f(y)$, so y and $f(y)$ are distinct points of T . Since T is Hausdorff, there are open sets U and V with $y \in U$, $f(y) \in V$, and $U \cap V = \emptyset$. Since f is continuous, $f^{-1}(V)$ is open, so $U \cap f^{-1}(V)$ is an open set containing y .

Now $U \cap f^{-1}(V)$ is disjoint from $\text{Fix}(f)$. To see this, suppose $x \in U \cap f^{-1}(V)$ and $x \in \text{Fix}(f)$. Since $x \in f^{-1}(V)$, $f(x) \in V$. But also $x \in U$ and $x \in \text{Fix}(f)$, so $f(x) \in U$. This implies $x \in U \cap V$, which is a contradiction. So there is no such x , and thus $U \cap f^{-1}(V)$ is disjoint from $\text{Fix}(f)$.

Thus, we have an open set $U \cap f^{-1}(V)$ such that $y \in U \cap f^{-1}(V) \subset T - \text{Fix}(f)$. Since y was an arbitrary point of $T - \text{Fix}(f)$, around every point in $T - \text{Fix}(f)$ we can find an open set contained in $T - \text{Fix}(f)$. Therefore, by the result of #2 on this worksheet, $T - \text{Fix}(f)$ is open, which means that $\text{Fix}(f)$ is closed.



$x \in \text{Fix}(f)$ means $f(x) = x$

This page is a bonus!

It's a remarkable proof, but we don't have time to discuss it in class.

5. For integers $a \neq 0$ and b , let $S(a, b) = \{an + b \mid n \in \mathbb{Z}\}$.

- (a) Show that the collection $\{S(a, b) \mid a, b \in \mathbb{Z}, a \neq 0\}$ is the basis for a topology on $\mathbb{Z}$.
This topology is called the **arithmetic sequence topology**.

(b) Show that the arithmetic sequence topology is Hausdorff.

(c) Show that the basis elements are both open and closed in this topology.

- (d) Use topology to prove that there are infinitely many primes as follows: First, define $Q \subset \mathbb{Z}$ as

$$Q = \bigcup_{p \text{ prime}} S(p, 0).$$

Assume that there are only finitely many primes, and explain why this implies that Q is closed. Thus, $\mathbb{Z} - Q$ must be open. What integers are elements of $\mathbb{Z} - Q$? Why is this a contradiction?