

TOPOLOGY DAY 11

20 Oct. 2026

DEFINITION: Topological spaces S and T are **homeomorphic** if there are continuous maps $f: S \rightarrow T$ and $g: T \rightarrow S$ such that $(f \circ g) = \text{id}_T$ and $(g \circ f) = \text{id}_S$.

Identity maps:
 $\text{id}_T(t) = t$ for all $t \in T$,
and similarly for S .

The maps S and T are **homeomorphisms**, and they are inverses: $f^{-1} = g$ and $g^{-1} = f$.

If S and T are homeomorphic, we write $S \cong T$.

↑ "congruent" symbol
in LaTeX: `\cong`

S^1 is not homeomorphic to $[0, 2\pi)$

↑ subspace topology from $\mathbb{R}^2$ ↑ subspace topology from $\mathbb{R}$

REASON: Remove an interior point from $[0, 2\pi)$ and then this space is disconnected.

It is not possible to obtain a disconnected space by removing a point from S^1 .

Homeomorphisms

MATH 348

1. Which of the following spaces do you think we should regard as topologically *the same*? What should “topologically the same” mean, exactly? (If a set below is a subset of $\mathbb{R}$, assume that it has its subspace topology from $\mathbb{R}$.)

(a) $(0, 1)$

(b) $[0, 1]$

(c) $(0, 1]$

(d) (a, b) for any $a < b \in \mathbb{R}$

(e) $[a, b]$ for any $a < b \in \mathbb{R}$

(f) S^1

(g) $\mathbb{R}$

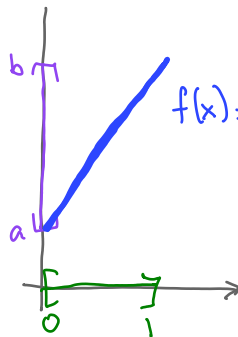
(h) $\mathbb{Q}$

(i) $[0, 2\pi)$

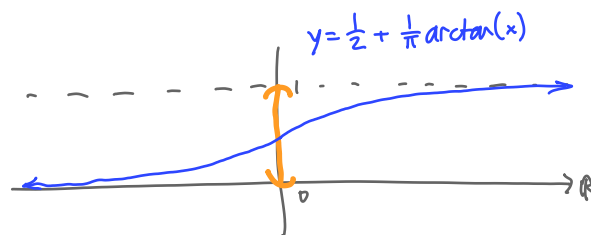
(j) $\mathbb{R} \cup \{\infty\}$

(k) $\mathbb{Q} \cap (0, 1)$

(l) Real line with a double point at 0 (from Homework 4 or Example 4.37 in the text)



$f(x) = x(b-a) + a$ is a homeomorphism between $[0, 1]$ and $[a, b]$



homeomorphism between $(0, 1)$ and $\mathbb{R}$

$f(x) = \frac{1}{2} \left(1 + \frac{x}{1+|x|} \right)$ is a homeomorphism between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$

Also: $\mathbb{Q} \cong \mathbb{Q} \cap [0, 1]$
Look up: Sierpinski's Theorem

2. Find three different topologies on the three-point set $\{a, b, c\}$, each consisting of five open sets, such that two of the topologies are homeomorphic, but the third is not homeomorphic to the other two.

HINT: match up points and open sets

↪ Lemma 5.4 in the text

3. Let $f: S \rightarrow T$ be a homeomorphism and $g: T \rightarrow U$ be another homeomorphism. Show that $g \circ f: S \rightarrow U$ is a homeomorphism.

↪ Proposition 5.16 in the text

4. Let S and T be homeomorphic spaces. Prove each of the following:

(a) If S is connected, then so is T .

(b) If S is compact, then so is T .

(c) If S is Hausdorff, then so is T .

5. Suppose $f: X \rightarrow Y$ is a continuous bijective function. Is f^{-1} necessarily continuous?

No! Consider the example from class:

$$f: [0, 2\pi) \rightarrow S^1 \text{ defined } f(x) = (\cos(x), \sin(x))$$

f is continuous, but f^{-1} is not continuous