Topology Day 13

27 October 2026

First: "natural maps" for disjoint unions and product spaces.

Disjoint Unions

For disjoint unions, the natural maps are **inclusions**.

Definition: i_S takes a point $x \in S$ to its corresponding (labeled) point in $S \amalg T$, and similarly for i_T.



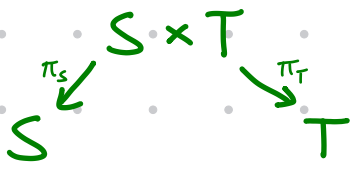
Are inclusion maps continuous?

Yes. Let $U \subset S \amalg T$ be open. Then $i_S^{-1}(U) = U \cap S$, which is open, showing that i_S is continuous. Similarly, i_T is continuous.

Product Spaces

For product spaces, the natural maps are **projections**.

Definition: $\pi_S(s,t) = s$ and $\pi_T(s,t) = t$



Are projection maps continuous?

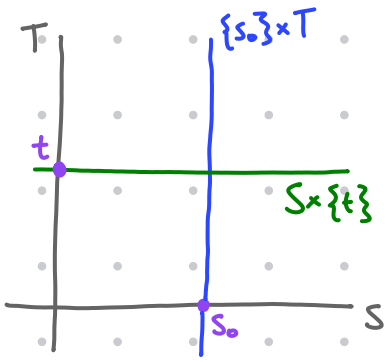
Yes. Let $U \subset S$ be open. Then $\pi_S^{-1}(U) = U \times T$, which is a basis element of the product topology and thus is open. This shows that π_S is continuous. Similarly, π_T is continuous.

THEOREM: If S and T are each connected spaces, then the product space $S \times T$ is connected

How to prove this? We might suppose that $S \times T$ has a separation, and then try to construct a separation of S or of T, but this is difficult. It's more productive to think about maps from $S \times T$ to a disconnected space.

PROOF: Suppose S and T are each connected. We will show that there is no continuous surjection from $S \times T$ to $S^0 = \{-1, 1\}$. (Note that if $S \times T$ were not connected, then such a continuous surjection would exist.)

Suppose that $f: S \times T \to S^0$ is continuous. Let $s_0 \in S$ be any point; then $\{s_0\} \times T$ is homeomorphic to T and hence is connected. So the restriction of f to $\{s_0\} \times T$ must be constant; assume without loss of generality that $f(s_0, t) = 1$ for all $t \in T$.



Similarly, for each $t \in T$, we can restrict f to $S \times \{t\}$, which is homeomorphic to S and thus connected. This implies f is constant on each such subset Since $(s_0, t) \in S \times \{t\}$ and $f(s_0, t) = 1$, it must be that $f(s, t) = 1$ for all $(s, t) \in S \times \{t\}$. The same argument applies for all $t \in T$, so $f(s, t) = 1$ for all $(s, t) \in S \times T$. Therefore, f is constant on $S \times T$, meaning that there is no continuous surjection $S \times T \to S^0$.