Topology Day 14

29 October 2026

Where is the "quotient" in a quotient space?

Consider the remainders after dividing integers.

For example, when dividing an integer by 5, the result is either 0, 1, 2, 3, or 4. Thus, we can define a map that sends each integer to its remainder after division by 5:

$$f: \mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z} = \{0,1,2,3,4\}$$

This map partitions $\mathbb{Z}$ into equivalence classes of integers with the same remainder mod 5.

$$\begin{aligned} 0, 5, 10, \ldots &\longmapsto 0 \\ 1, 6, 11, \ldots &\longmapsto 1 \\ 2, 7, 12, \ldots &\longmapsto 2 \\ 3, 8, 13, \ldots &\longmapsto 3 \\ 4, 9, 14, \ldots &\longmapsto 4 \end{aligned}$$

In abstract algebra, the notation $\mathbb{Z}/5\mathbb{Z}$ denotes the quotient group of integers 0, 1, 2, 3, 4 with arithmetic mod 5 (only consider remainders after division by 5).

Quotient Spaces in Topology

We can **quotient by a subspace**.

Example from the text:

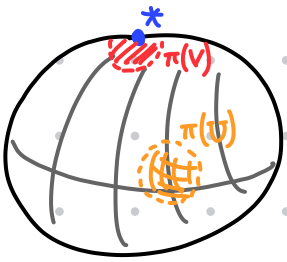
closed disc D^2 in the plane $\mathbb{R}^2$



(an open set containing the boundary of the disc)

collapse the boundary of the disc to a point

π



If X is a topological space and $A \subset X$ is a subspace, then the **quotient space** X/A is the set $(X-A) \amalg \{*\}$, where we have replaced A with a new point denoted $*$.

A set $U \subset (X-A) \amalg \{*\}$ is open if and only if $\pi^{-1}(U)$ is open in X, where $\pi : X \to (X-A) \amalg \{*\}$ is the map defined $\pi(x) = x$ if $x \notin A$ and $\pi(x) = *$ if $x \in A$.

More generally, we can **quotient by an equivalence relation**.

An equivalence relation is a binary relation $\sim$ on a set X that is:

- reflexive: $a \sim a$ for all $a \in X$,
- symmetric: if $a \sim b$, then $b \sim a$,
- transitive: if $a \sim b$ and $b \sim c$, then $a \sim c$.

If $\sim$ is an equivalence relation on X, then we can form the quotient set $X/\sim$ consisting of equivalence classes of elements of X under $\sim$. The natural function π takes each element of X to its equivalence class. The **quotient space** is the set $X/\sim$, with a subset $U \subset X/\sim$ open if and only if $\pi^{-1}(U)$ is open.

Quotient Spaces

MATH 348

1. Describe each quotient space given by $X/\sim$.

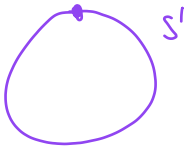
(a) $X = \{a, b, c, d, e\}$ with the topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. Define an equivalence relation on X by $a \sim b$ and $c \sim d \sim e$.

$X/\sim$ is a two-point set containing $A = \{a, b\}$ and $B = \{c, d, e\}$.

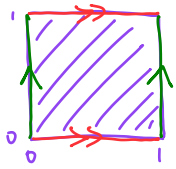
Open sets: $\{\emptyset, A, \{A, B\}\}$ $\quad X \xrightarrow{\pi} X/\sim$

(b) $X = [0, 1]$ with $x \sim x'$ if $x - x' \in \mathbb{Z}$

$0 \sim 1$

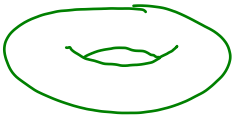


(c) $X = [0, 1]^2$ with $(x, y) \sim (x', y')$ if $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$





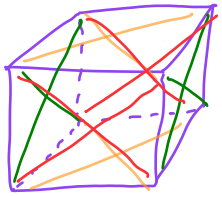
torus T^2



$T^2 \cong X/\sim \cong S^1 \times S^1$

$T^2 \not\cong X = [0,1]^2$

(d) $X = [0, 1]^3$ with $(x, y, z) \sim (x', y', z')$ if $x - x' \in \mathbb{Z}$, $y - y' \in \mathbb{Z}$, and $z - z' \in \mathbb{Z}$



3-torus

$T^3 \cong S^1 \times S^1 \times S^1$

Not a 2-holed torus



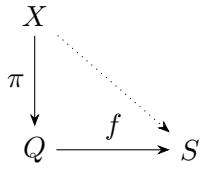
2. Let $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$ and $Y = [0, 2] \subset \mathbb{R}$. Define $f : X \to Y$ as

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ x - 1 & \text{if } x \in [2, 3] \end{cases}.$$

Do you think that the map f allows us to consider Y to be a quotient space of X? Why or why not?

3. Suppose X is a topological space with equivalence relation $\sim$. Let Q be the quotient space $X/\sim$, and let $\pi : X \to Q$ be the projection (quotient) map.

(a) If S is any other topological space and $f : Q \to S$ is continuous, explain why there exists a map $g : X \to S$ such that the following diagram commutes.



(b) If $g : X \to S$ is any continuous function such that $g(x) = g(y)$ whenever $x \sim y$, explain why there is a continuous function $f : Q \to S$ such that the following diagram commutes.

