Topology Day 15

3 November 2026

Definition: Let X and Y be topological spaces and $f: X \to Y$ be a surjective map. Then f is a **quotient map** if each $U \subset Y$ is open in Y if and only if $f^{-1}(U)$ is open in X.

Homotopy: Interpolation between functions

Definition: Two maps $f, g: S \to T$ are **homotopic** if there is a continuous function $F: S \times [0,1] \to T$ such that $F(s,0) = f(s)$ and $F(s,1) = g(s)$ for all $s \in S$.

In this case, F is a **homotopy** between f and g, and we write $f \simeq g$.

↑ symbol $\simeq$ is `\simeq` in LaTeX.

Quotient Spaces

MATH 348

1. Describe each quotient space given by $X/\sim$.

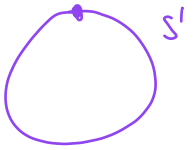
(a) $X = \{a, b, c, d, e\}$ with the topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. Define an equivalence relation on X by $a \sim b$ and $c \sim d \sim e$.

$X/\sim$ is a two-point set containing $A = \{a, b\}$ and $B = \{c, d, e\}$.

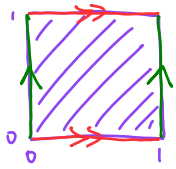
Open sets: $\{\emptyset, A, \{A, B\}\}$ $\quad X \xrightarrow{\pi} X/\sim$

(b) $X = [0, 1]$ with $x \sim x'$ if $x - x' \in \mathbb{Z}$

$0 \sim 1$

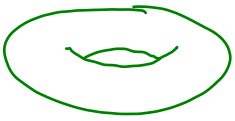


(c) $X = [0, 1]^2$ with $(x, y) \sim (x', y')$ if $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$





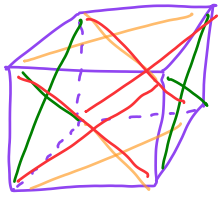
torus T^2



$T^2 \cong X/\sim \cong S^1 \times S^1$

$T^2 \not\cong X = [0,1]^2$

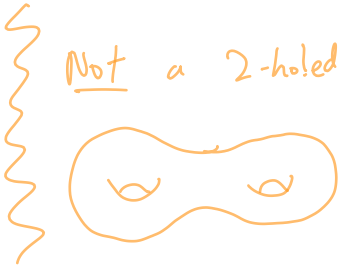
(d) $X = [0, 1]^3$ with $(x, y, z) \sim (x', y', z')$ if $x - x' \in \mathbb{Z}$, $y - y' \in \mathbb{Z}$, and $z - z' \in \mathbb{Z}$



3-torus

$T^3 \cong S^1 \times S^1 \times S^1$

Not a 2-holed torus



2. Let $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$ and $Y = [0, 2] \subset \mathbb{R}$. Define $f : X \to Y$ as

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ x - 1 & \text{if } x \in [2, 3] \end{cases}.$$

Do you think that the map f allows us to consider Y to be a quotient space of X? Why or why not?

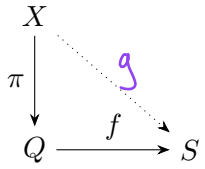
Equivalence relation: $1 \sim 2$ in X. Then $Y = X/\sim$.

Subspace: $Y \cong X/\{1, 2\}$

Theorem 5.62 in Crossley

3. Suppose X is a topological space with equivalence relation $\sim$. Let Q be the quotient space $X/\sim$, and let $\pi : X \to Q$ be the projection (quotient) map.

(a) If S is any other topological space and $f : Q \to S$ is continuous, explain why there exists a map $g : X \to S$ such that the following diagram commutes.

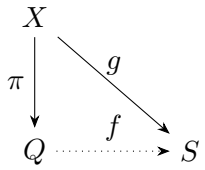


Let $g = f \circ \pi$.

Then $g(x) = (f \circ \pi)(x)$ for all $x \in X$,

and g is continuous as a composition of continuous functions.

(b) If $g : X \to S$ is any continuous function such that $g(x) = g(y)$ whenever $x \sim y$, explain why there is a continuous function $f : Q \to S$ such that the following diagram commutes.



Define $f: Q \to S$ by $f(E) = g(x)$ if E is the equivalence class containing x. Note that if $y \in E$, then $y \sim x$, so $g(x) = g(y)$, and thus $f(E) = g(y)$. Thus, it does not matter which element of E we choose to define $f(E)$, so f is well defined. Hence, $g = f \circ \pi$.

To show f is continuous, let U be an open set in S. Then:

$$g^{-1}(U) = (f \circ \pi)^{-1}(U) = \pi^{-1}\left(f^{-1}(U)\right)$$

Since g is continuous, $g^{-1}(U)$ is open. That is, $\pi^{-1}\left(f^{-1}(U)\right)$ is open. Since Q has the quotient topology, π^{-1} of a set is open if and only if that set is open, so $\pi^{-1}\left(f^{-1}(U)\right)$ being open implies $f^{-1}(U)$ is open. Therefore, f is continuous.