# Topology, Day 16

5 November 2026

## Homotopic Functions

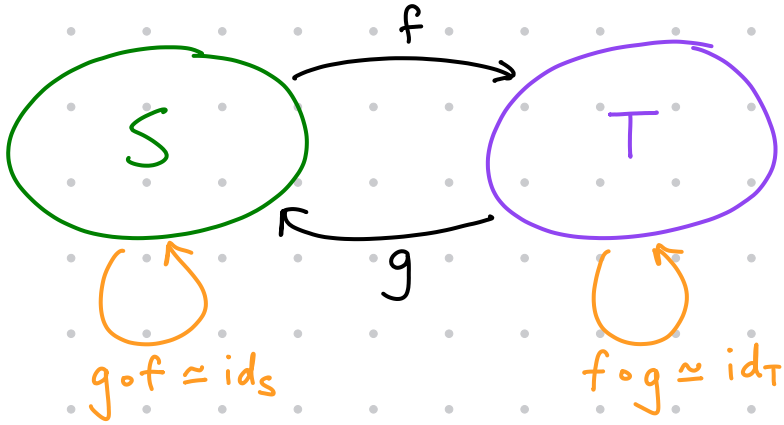
Two maps $f, g: S \to T$ are **homotopic** if there is a continuous function $F: S \times [0,1] \to T$ such that $F(s,0) = f(s)$ and $F(s,1) = g(s)$ for all $s \in S$.

In this case, $F$ is a **homotopy** between $f$ and $g$, and we write $f \simeq g$.

↳ symbol $\simeq$ is `\simeq` in LaTeX.

## Homotopy Equivalent Topological Spaces

Topological spaces $S$ and $T$ are **homotopy equivalent** if there are continuous maps $f: S \to T$ and $g: S \to T$ such that $g \circ f$ is homotopic to the identity on $S$ and $f \circ g$ is homotopic to the identity on $T$. The maps $f$ and $g$ are then called **homotopy equivalences** and we write $S \simeq T$.



# Homotopy

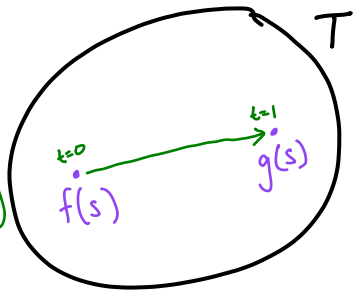
MATH 348

**1.** Suppose $T$ is convex and $S$ is any topological space. Why are every two maps $f, g : S \to T$ homotopic?

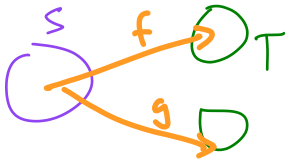
Since $T$ is convex, we can use a straight-line homotopy between any $f(s)$ and $g(s)$:

$$F(s,t) = (1-t)f(s) + t\,g(s) = f(s) + t(g(s)-f(s))$$

$F: S \times I \to T$




**2.** Give an example of spaces $S$ and $T$ with non-homotopic functions $f, g : S \to T$.




$S = T$ is disconnected

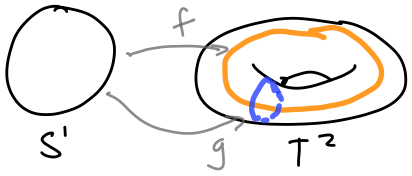
$f: S \to S$ identity $f(s) = s$

$g: S \to S$ swaps two connected components




$S = S^1$, $T = T^2$ torus

Circle




**3.** Justify each of the following to show that homotopy is an equivalence relation on functions between two given topological spaces:

(a) Homotopy is reflexive. (Lemma 6.6 in Crossley)

$f \simeq f$

(b) Homotopy is symmetric. (Lemma 6.7 in Crossley)

If $f \simeq g$, then $g \simeq f$.

(c) Homotopy is transitive. (Lemma 6.8 in Crossley)

If $f \simeq g$ and $g \simeq h$, then $f \simeq h$.

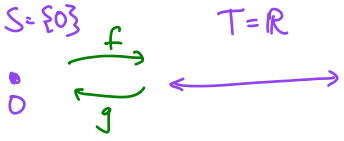
$\{0\}$ and $\mathbb{R}$ are NOT homeomorphic, but they are homotopy equivalent

**4.** Explain which of the following pairs of spaces are homotopy equivalent.

Contractible Spaces

(a) The single point space $\{0\}$ and $\mathbb{R}$

$\{0\} \simeq \mathbb{R}$




maps:

$f: \{0\} \to \mathbb{R}$ defined $f(0) = \frac{\pi^2}{6}$ (any real number)

$g: \mathbb{R} \to \{0\}$ defined $g(x) = 0$

compositions: $(g \circ f)(0) = 0$ so $g \circ f = id_S$ AND $(f \circ g)(x) = f(0) = \frac{\pi^2}{6}$

Since $\mathbb{R}$ is convex, $f \circ g \simeq id_{\mathbb{R}}$

(b) The closed interval $[0, 1]$ and the single point space $\{0\}$ $f \circ g$ is constant

$[0,1] \simeq \{0\}$

$f: [0,1] \to \{0\}$ defined $f(x) = 0$

$g: \{0\} \to [0,1]$ defined $g(0) = \frac{1}{2}$

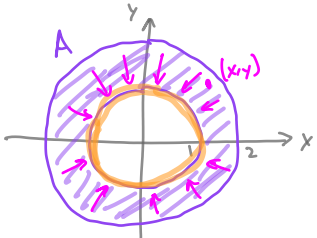
(c) The open interval $(0, 1)$ and the single point space $\{0\}$

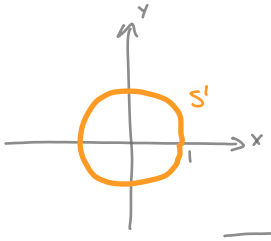
$(0,1) \simeq \{0\}$ Same maps

NOT CONTRACTIBLE

(d) The annulus $A = \{(x, y) \in \mathbb{R} \mid 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ and the circle $S^1$

$S^1 \simeq A$




$f: S' \to A$ defined $f(x,y) = (x,y)$

$g: A \to S'$ defined $g(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$

CHECK: $g \circ f = id_{S'}$

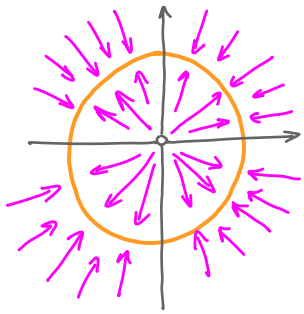
$(f \circ g)(x) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$

$f \circ g \simeq id_A$

(e) The punctured plane $\mathbb{R}^2 - \{(0,0)\}$ and the circle $S^1$

Yes!

$\mathbb{R}^2 - \{(0,0)\} \simeq S^1$



Radially project $\mathbb{R}^2 - \{(0,0)\}$ onto $S^1$.

(f) $S^1$ and $S^0$

Not homotopy equivalent.

See Proposition 6.20 in Crossley.