Topology, Day 17

10 November 2026

Homotopic Functions

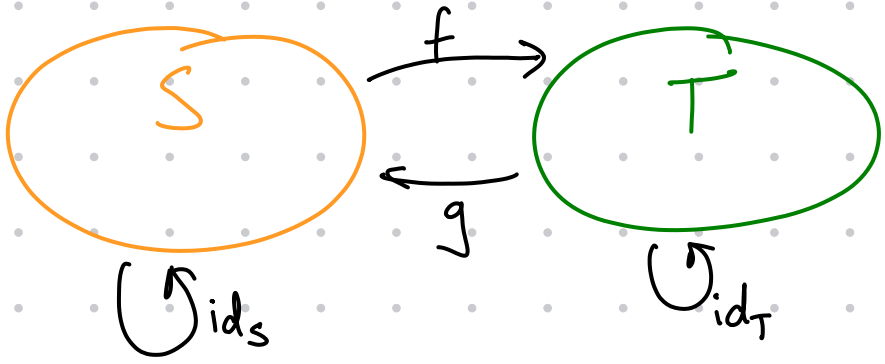
Definition: Two maps $f, g : S \to T$ are **homotopic** if there is a continuous function

$$F : S \times [0,1] \longrightarrow T$$

such that $F(s,0) = f(s)$ for all $s \in S$ and $F(s,1) = g(s)$ for all $s \in S$. In this case, F is a **homotopy** between f and g, and we write $f \simeq g$.

Homotopy Equivalent Spaces

Definition: Two topological spaces S, T are said to be **homotopy equivalent** if there are continuous maps $f : S \to T$ and $g : T \to S$ such that $g \circ f$ is homotopic to the identity on S and $f \circ g$ is homotopic to the identity on T. The maps f and g are then **homotopy equivalences**. If S and T are homotopy equivalent, then we write $S \simeq T$.



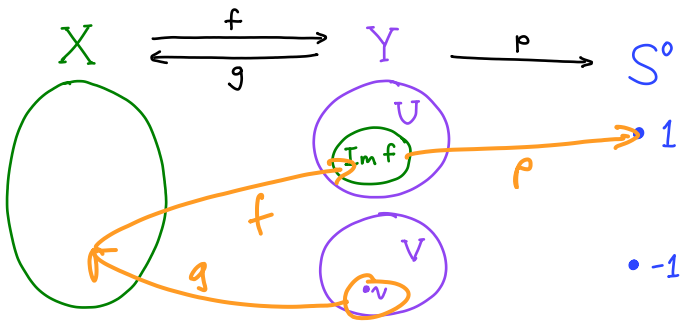
Homotopy and Connectivity

MATH 348

Theorem: If X is a connected topological space and Y is a disconnected topological space, then X and Y are not homotopy equivalent.

Use the following diagram and outline to complete a proof of the previous theorem.

Let X is a connected topological space and Y is a disconnected topological space. Suppose $X \simeq Y$. Consider the maps in the following diagram.



1. Why does there exists $F : Y \times I \to Y$ such that $F(y, 0) = f(g(y))$ and $F(y, 1) = y$?

Since $X \simeq Y$, $f \circ g \simeq id_Y$. So there is a function $F: Y \times I \to Y$ such that $F(y,0) = f(g(y))$ and $F(y,1) = y$.

2. Why does there exist $p : Y \to S^0$ with $p(y) = 1$ if $y \in U$ and $p(y) = -1$ if $y \in V$?

Since Y is disconnected, there exists disjoint nonempty open sets U and V such that $Y = U \cup V$. Define $p: Y \to S^0$ such that $p(y) = 1$ if $y \in U$ and $p(y) = -1$ if $y \in V$. So p is continuous and surjective.

3. Define $h : [0, 1] \to S^0$ by $h(t) = p(F(v, t))$. What can you say about h?

Since X is connected, without loss of generality assume $im(f) \subset U$.

Since V is not empty, there exists some $v \in V$. Then define $h(t) = p(F(v,t))$.

h is continuous, as a composition of continuous functions.

4. Why is $h(0) = 1$ and $h(1) = -1$? What does this allow you to conclude?

Then: $h(0) = p(F(v,0)) = p(f(g(v))) = 1$ since $f(g(v)) \in U$.

$h(1) = p(F(v,1)) = p(v) = -1$ since $v \in V$.

$F(v,1) = id_Y$

So h is a continuous surjection from $[0,1]$ to S^0, which is a contradiction.

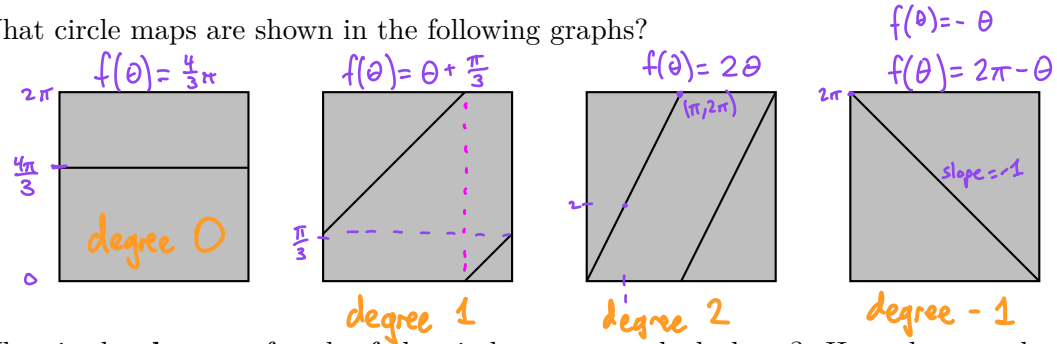
Now that you understand all of the steps in the proof, write the proof nicely in paragraph form on your own paper.

Therefore, X and Y are not homotopy equivalent

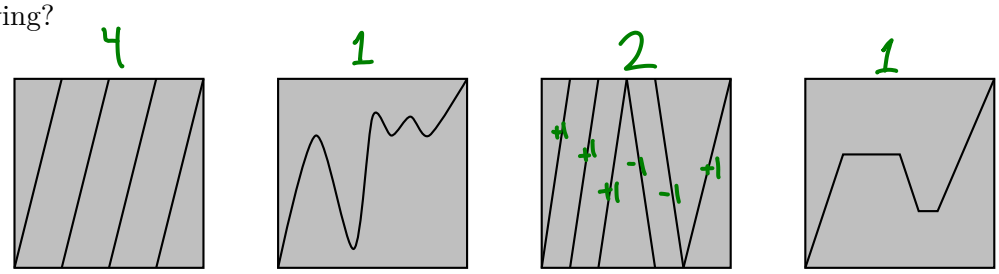
Circle Maps

MATH 348

1. What circle maps are shown in the following graphs?



2. What is the **degree** of each of the circle maps graphed above? How about each of the following?



3. Sketch the graph of a circle map with degree 5. Then sketch a circle map with degree -2.



4. Prove the statement: If a circle map $f : S^1 \to S^1$ has a nonzero degree, then it is surjective.

CONTRAPOSITIVE: If f is not surjective, then there is some point on S^1 that is not in the image of f. Then f can't wrap around S^1, so f must have zero degree.

5. Under what condition, do you think, are two circle maps are homotopic? Why?

$f \simeq g$ if and only if f and g have the same degree

REFERENCE: Crossley, Sec. 6.3
Adams and Franzosa, Sec. 9.6

6. Is the circle S^1 contractible? Can you prove your answer? No, S^1 is not contractible.

PROOF BY CONTRADICTION: Suppose $S^1 \simeq \{0\}$, with homotopy equivalences $f: S^1 \to \{0\}$ and $g: \{0\} \to S^1$. This means that $(g \circ f)(\theta) = g(0)$ for all $\theta \in S^1$, so $g \circ f: S^1 \to S^1$ is a constant function, meaning $g \circ f$ is a circle map of degree zero. But the identity map on S^1 has degree 1, so $g \circ f$ cannot be homotopic to the identity map, contradicting the statement that f and g are homotopy equivalences. So, S^1 is not homotopy equivalent to $\{0\}$.