Homework 6

MATH 348

due at 11pm on Tuesday, November 3, 2026

Solve the following problems and communicate your solutions clearly using complete sentences. Your proofs may rely on definitions and theorems stated in the text or given in class.

Remember what the syllabus says about appropriate collaboration, and document what sources you use and what assistance you receive as you work on this homework. *You must include an acknowledgment of resources and collaboration as part of your homework solutions.*

For this homework, you must type your solutions to all of the problems in LaTeX. You may include hand-drawn diagrams in your solutions. Make sure your solutions are easy to read, in order, and clearly labeled. Upload a single file containing your solutions to the Homework 6 assignment on Moodle.

Some of the problems will be graded in detail, and the rest will be graded for completion.

1. (8 points) Let X and Y be topological spaces, and let f be a map from X to Y. The map f is an **open map** if for each open set $U \subset X$, the set $f(U)$ is open in Y. Similarly, f is a **closed map** if for each ~~open~~ closed set $V \subset X$, the set $f(V)$ is closed in Y.

(a) Consider the inclusion maps from topological spaces S and T to their disjoint union $S \amalg T$:

$$i_S : S \to S \amalg T \qquad \text{and} \qquad i_T : T \to S \amalg T.$$

These maps are defined such that i_S takes a point $x \in S$ to its corresponding (labeled) copy in $S \amalg T$, and similarly for i_T.

Are inclusion maps open maps? Are inclusion maps closed maps? Explain.

(b) Consider the projection maps from a product space $S \times T$ to the individual spaces S and T:

$$\pi_1 : S \times T \to S \qquad \text{and} \qquad \pi_2 : S \times T \to T.$$

These maps are defined $\pi_S(s,t) = s$ and $\pi_T(s,t) = t$.

Are projection maps open maps? Are projection maps closed maps? Explain.

2. (6 points) Prove that if $S \times T$ is compact, then S and T are both compact.

3. (6 points) Let $f : X \to Y$ be a function. The *graph* of f is the subset of $X \times Y$ given by $G = \{(x, f(x)) \mid x \in X\}$. Show that if f is continuous and Y is Hausdorff, then G is closed in $X \times Y$.

4. (5 points) In each of the following cases, describe the resulting quotient space. A sketch may be helpful but is not a complete answer. Make sure to specify what are the points in the quotient space and how they are connected. You do not need to precisely state all of the open sets.

(a) The disk with its boundary points identified with each other to form a single point.

(b) The interval $[0, 4]$, as a subspace of $\mathbb{R}$, with integer points identified with each other.

(c) The real line $\mathbb{R}$ with $[-1, 1]$ collapsed to a point.

(d) The plane $\mathbb{R}^2$ with the circle S^1 collapsed to a point.

(e) The sphere S^2 with the equator collapsed to a point.

5. (6 points) Let $X = \mathbb{R}$ and define an equivalence relation on X by $x \sim y$ if $\lfloor x \rfloor = \lfloor y \rfloor$. Here, $\lfloor x \rfloor$ denotes the *floor* of x, the greatest integer less than or equal to x.

(a) Show that $\sim$ defines an equivalence relation on $\mathbb{R}$.

(b) What are the elements of the quotient space $X/\sim$?

(c) What are the open sets in the quotient space $X/\sim$?

6. (6 points) Let $X = \mathbb{R}^2$ and define an equivalence relation on X by $(x, y) \sim (z, w)$ if

$$x^2 + y^2 = z^2 + w^2$$

(a) Show that $\sim$ defines an equivalence relation on $\mathbb{R}^2$.

(b) What are the elements of the quotient space $X/\sim$?

(c) What are the open sets in the quotient space $X/\sim$?

7. (4 points) Show that the topology defined on a quotient space (in Section 5.4 of our text) is really a topology. (That is, show that it satisfies the four axioms of a topology.)