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So far: transition matrix $\longrightarrow$ steady-state dist

today: Given a steady-state distribution, can we find a transition matrix?

EXAMPLE: steady-state vector $\vec{v} = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle^T$

States: S_1, S_2, S_3 with probabilities $p_1 = \frac{1}{6}, p_2 = \frac{1}{3}, p_3 = \frac{1}{2}$

lowercase p

IDEA: ① Select a 3×3 symmetric proposal matrix Q

let $Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

$P = \begin{bmatrix} & ? & \end{bmatrix}$

row i, col j

② Given states S_i and S_j with $i \neq j$, define transition probability $P_{i,j} = \text{Prob.}[j \to i]$ by:

- If $p_j < p_i$, then let $P_{i,j} = Q_{i,j} = \frac{1}{3}$
- If $p_j \geq p_i$, then let $P_{i,j} = \frac{p_i}{p_j} Q_{i,j}$

③ Define $P_{j,j} = 1 - \sum_{k \neq j} P_{k,j}$ $\leftarrow$ ensure that columns sum to 1

Back to our example:

- Column $j = 1$:

$i=2$: $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{3}$: $p_1 < p_2$, so $P_{2,1} = Q_{2,1} = \frac{1}{3}$

$i=3$: $p_1 = \frac{1}{6}$, $p_3 = \frac{1}{2}$: $p_1 < p_3$, so $P_{3,1} = Q_{3,1} = \frac{1}{3}$

$$\boxed{P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}}$$

- Column: $j=2$:

  $i=1$: $p_2 = \frac{1}{3}$, $p_1 = \frac{1}{6}$: $p_2 > p_1$ so $P_{1,2} = \frac{p_1}{p_2} Q_{1,2} = \frac{\frac{1}{6}}{\cancel{\frac{1}{3}}} \cdot \cancel{\frac{1}{3}} = \frac{1}{6}$

  $i=3$: $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{2}$ $p_2 < p_3$ so $P_{3,2} = Q_{3,2} = \frac{1}{3}$

- Column: $j=3$:

  $i=1$: $p_3 = \frac{1}{2}$, $p_1 = \frac{1}{6}$, $p_3 > p_1$ so $P_{1,3} = \frac{p_1}{p_3} Q_{1,3} = \frac{\frac{1}{6}}{\frac{1}{2}} \cdot \frac{1}{3} = \frac{2}{18} = \frac{1}{9}$

  $i=2$: $p_3 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, $p_3 > p_2$ so $P_{2,3} = \frac{p_2}{p_3} Q_{2,3} = \frac{\frac{1}{3}}{\frac{1}{2}} \cdot \frac{1}{3} = \frac{2}{9}$

PRACTICE:

① Find a transition matrix with steady-state distribution.

$$\vec{v} = \left\langle \frac{3}{10}, \frac{1}{2}, \frac{1}{5} \right\rangle$$

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \longrightarrow P = \begin{bmatrix} \frac{4}{9} & \frac{1}{5} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{9} & \frac{2}{15} & \frac{1}{3} \end{bmatrix}$$

② Find another, different matrix with the same steady-state distribution.

$$Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \longrightarrow P = \begin{bmatrix} \frac{13}{18} & \frac{1}{10} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{15} & \frac{2}{3} \end{bmatrix}$$

GOAL: Simulate a Markov chain on a large state space without having to store a transition matrix

We will know long-term frequencies of each state'

EXAMPLE: frequencies $\vec{V} = \langle 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \rangle$ (ten states)

Simulate chain of 1,000,000 steps

want distribution like: